

# Solving the Metric Similarities Problem

Recall that our  $q$  by  $s$  matrix of stimuli coordinates is:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} & \cdot & \cdot & \cdot & \mathbf{z}_{1s} \\ \mathbf{z}_{21} & \mathbf{z}_{22} & \cdot & \cdot & \cdot & \mathbf{z}_{2s} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \mathbf{z}_{q1} & \mathbf{z}_{q2} & \cdot & \cdot & \cdot & \mathbf{z}_{qs} \end{bmatrix}$$

And the  $q$  by  $q$  matrix of squared distances between the  $q$  stimuli is:

$$\mathbf{D}_z = \begin{bmatrix} \sum_{k=1}^s (\mathbf{z}_{1k} - \mathbf{z}_{1k})^2 & \sum_{k=1}^s (\mathbf{z}_{1k} - \mathbf{z}_{2k})^2 & \cdot & \cdot & \cdot & \sum_{k=1}^s (\mathbf{z}_{1k} - \mathbf{z}_{qk})^2 \\ \sum_{k=1}^s (\mathbf{z}_{2k} - \mathbf{z}_{1k})^2 & \sum_{k=1}^s (\mathbf{z}_{2k} - \mathbf{z}_{2k})^2 & \cdot & \cdot & \cdot & \sum_{k=1}^s (\mathbf{z}_{2k} - \mathbf{z}_{qk})^2 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum_{k=1}^s (\mathbf{z}_{qk} - \mathbf{z}_{1k})^2 & \sum_{k=1}^s (\mathbf{z}_{qk} - \mathbf{z}_{2k})^2 & \cdot & \cdot & \cdot & \sum_{k=1}^s (\mathbf{z}_{qk} - \mathbf{z}_{qk})^2 \end{bmatrix}$$

If there is no error then the solution is:

1. Double-Center  $\mathbf{D}_z$

$$\mathbf{Y} = \mathbf{Z}^* \mathbf{Z}^{*'} = \begin{bmatrix} \mathbf{z}_{11} - \bar{\mathbf{z}}_1 & \mathbf{z}_{12} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{1s} - \bar{\mathbf{z}}_s \\ \mathbf{z}_{21} - \bar{\mathbf{z}}_1 & \mathbf{z}_{22} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{2s} - \bar{\mathbf{z}}_s \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{z}_{q1} - \bar{\mathbf{z}}_1 & \mathbf{z}_{q2} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{qs} - \bar{\mathbf{z}}_s \end{bmatrix} \begin{bmatrix} \mathbf{z}_{11} - \bar{\mathbf{z}}_1 & \mathbf{z}_{12} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{1s} - \bar{\mathbf{z}}_s \\ \mathbf{z}_{21} - \bar{\mathbf{z}}_1 & \mathbf{z}_{22} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{2s} - \bar{\mathbf{z}}_s \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{z}_{q1} - \bar{\mathbf{z}}_1 & \mathbf{z}_{q2} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{qs} - \bar{\mathbf{z}}_s \end{bmatrix}'$$

2. Compute the eigenvalue-eigenvector decomposition of  $\mathbf{Y}$ :

$$\mathbf{Y} = \mathbf{U} \mathbf{A} \mathbf{U}'$$

3. Set  $\mathbf{Z} = \mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}$

Note that, without loss of generality you can assume that  $\bar{\mathbf{z}} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{z}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$

If there is error then the solution is somewhat harder.

1. Define  $\mathbf{d}_{jm}^* = \mathbf{d}_{jm} + \boldsymbol{\varepsilon}_{jm} = \sqrt{\sum_{k=1}^s (\mathbf{z}_{jk} - \mathbf{z}_{mk})^2} + \boldsymbol{\varepsilon}_{jm}$  and set the squared-error loss function to:

$$\mu = \sum_{j=1}^q \sum_{m=1}^q \boldsymbol{\varepsilon}_{jm}^2 = \sum_{j=1}^q \sum_{m=1}^q (\mathbf{d}_{jm}^* - \mathbf{d}_{jm})^2 \quad (1)$$

2. The first derivatives are:

$$\begin{aligned} \frac{\partial \mu}{\partial \mathbf{z}_{jk}} &= 2 \sum_{m=1}^q \left\{ (\mathbf{d}_{jm}^* - \mathbf{d}_{jm}) \left( -\frac{1}{2} \right) \left[ \sum_{k=1}^s (\mathbf{z}_{jk} - \mathbf{z}_{mk})^2 \right]^{-\frac{1}{2}} (2[\mathbf{z}_{jk} - \mathbf{z}_{mk}]) \right\} \\ &= -2 \sum_{m=1}^q \left\{ \left( \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} - 1 \right) (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \right\} \quad (2) \end{aligned}$$

3. Setting equal to zero and solving for  $\mathbf{z}_{jk}$ :

$$\sum_{m=1}^q \left[ \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \right] - \sum_{m=1}^q (\mathbf{z}_{jk} - \mathbf{z}_{mk}) = 0$$

Rearranging:

$$-q\mathbf{z}_{jk} + \sum_{m=1}^q \left[ \mathbf{z}_{mk} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \right] = 0$$

Therefore:

$$\hat{\mathbf{z}}_{jk} = \frac{1}{\mathbf{q}} \sum_{m=1}^{\mathbf{q}} \left[ \mathbf{z}_{mk} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \right] \quad (3)$$

Note that the solution is in the form:

$$\mathbf{z} = \mathbf{f}(\mathbf{y}, \mathbf{z})$$

That is, the solution for  $\mathbf{z}$  is a value such that when it is plugged into  $\mathbf{f}(\mathbf{y}, \mathbf{z})$  it produces itself!

Define:

$$\mathbf{z}_{jkm} = \mathbf{z}_{mk} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \quad (4)$$

So that equation (3) can be re-written as:

$$\hat{\mathbf{z}}_{jk} = \frac{1}{\mathbf{q}} \sum_{m=1}^{\mathbf{q}} \mathbf{z}_{jkm} \quad (5)$$

Using equation (4), note that the *point*  $\mathbf{z}_{j,m}$  is:

$$\mathbf{z}_{j,m} = \begin{bmatrix} \mathbf{z}_{m1} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{j1} - \mathbf{z}_{m1}) \\ \mathbf{z}_{m2} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{j2} - \mathbf{z}_{m2}) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{z}_{ms} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{js} - \mathbf{z}_{ms}) \end{bmatrix} = \mathbf{z}_m + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_j - \mathbf{z}_m) \quad (6)$$

Where  $\mathbf{z}_j = \begin{bmatrix} \mathbf{z}_{j1} \\ \mathbf{z}_{j2} \\ \cdot \\ \cdot \\ \mathbf{z}_{js} \end{bmatrix}$  and  $\mathbf{z}_m = \begin{bmatrix} \mathbf{z}_{m1} \\ \mathbf{z}_{m2} \\ \cdot \\ \cdot \\ \mathbf{z}_{ms} \end{bmatrix}$  are points and  $\frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}}$  is a scalar. *Equation (6) is*

*the basic equation of a line that passes through  $\mathbf{z}_j$  and  $\mathbf{z}_m$ !* The general formula for a line equation is:

$$\mathbf{Y}(t) = \mathbf{A} + t(\mathbf{B} - \mathbf{A}) \quad (7)$$

Where  $\mathbf{A}$  and  $\mathbf{B}$  are points and  $t$  is a scalar. Note that if  $0 < t < 1$  then equation (7) defines a line that runs between points  $\mathbf{A}$  and  $\mathbf{B}$ .

Once specific values are plugged into equation (6) then the solution for the point

$\hat{\mathbf{z}}_j = \begin{bmatrix} \hat{\mathbf{z}}_{j1} \\ \hat{\mathbf{z}}_{j2} \\ \cdot \\ \cdot \\ \hat{\mathbf{z}}_{js} \end{bmatrix}$  is simply *the centroid of the  $q$   $\mathbf{z}_{j,m}$  points!*

Finally, note that the squared distance between the points  $\mathbf{z}_j$  and  $\mathbf{z}_{j,m}$  is:

$$\begin{aligned} \sum_{k=1}^s (\mathbf{z}_{jk} - \mathbf{z}_{jkm})^2 &= \sum_{k=1}^s \left[ \mathbf{z}_{jk} - \left( \mathbf{z}_{mk} + \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \right) \right]^2 = \\ \sum_{k=1}^s \left[ (\mathbf{z}_{jk} - \mathbf{z}_{mk}) \left( 1 - \frac{\mathbf{d}_{jm}^*}{\mathbf{d}_{jm}} \right) \right]^2 &= \frac{(\mathbf{d}_{jm} - \mathbf{d}_{jm}^*)^2}{\mathbf{d}_{jm}^2} \left( \sum_{k=1}^s (\mathbf{z}_{jk} - \mathbf{z}_{mk})^2 \right) = \mathbf{\epsilon}_{jm}^2 \quad (8) \end{aligned}$$

So that the squared error is represented directly on the  $s$ -dimensional hyperplane (see below).

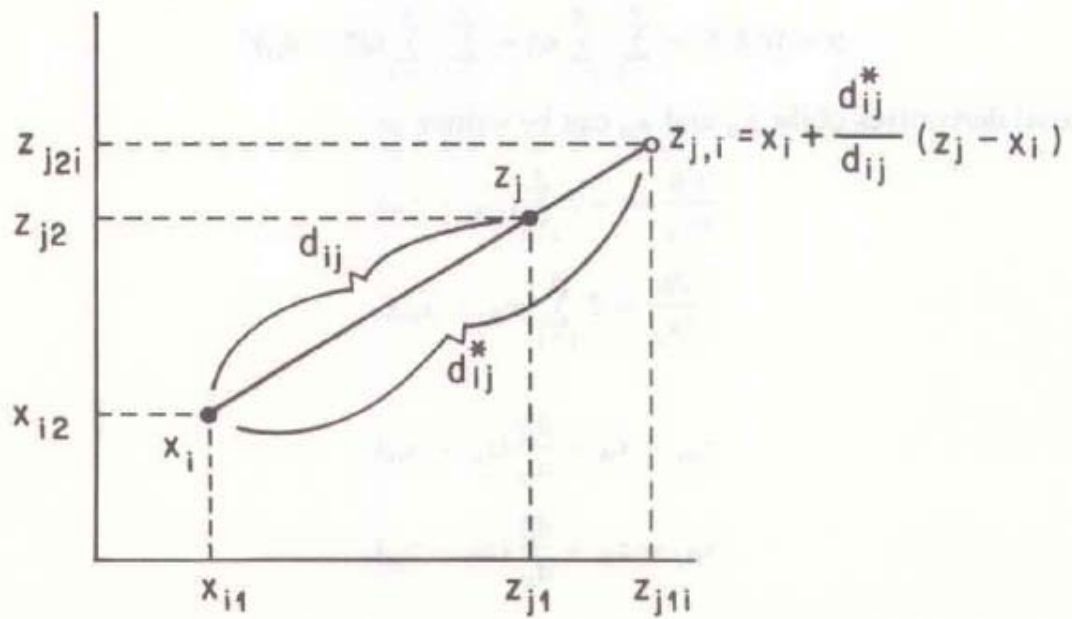


FIGURE 1  
Parametric equation of a straight line.